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Partial Dimensional Sequences and Percolation

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We discuss a new mathematical approach of describing fractal lattices by means of transfer matrices of fractals (TMFs). These matrices have interesting mathematical properties. Possible physical applications of the TMFs are briefly indicated.

KEY WORDS: Fractals; transfer matrices; percolation; clusters; renormalization group.

In this paper we discuss a new method of studying the geometry of complicated, nonhomogeneous fractals. This method has been recently reported in some detail[1] and some possible physical applications have been indicated. It consists of constructing geometrical transfer matrices (TMFs), which describe the iterative decoration processes involved in generating fractal lattices. Here we shall demonstrate this method by two simple examples. We then mention briefly some of the general properties of TMFs, and their physical relevance.

Let us consider a Sierpinski carpet[2,3] with a rescaling factor $b = 5$, where one (the central) subsquare eliminated at each iteration. The fractal dimensionality, D_1, of this lattice is

$$D_1 = \log 24/\log 5 \simeq 1.975 \tag{1}$$

Now, in addition to eliminating the central square, we also eliminate its boundaries. One such construction iteration is shown in Fig. 1. Starting with

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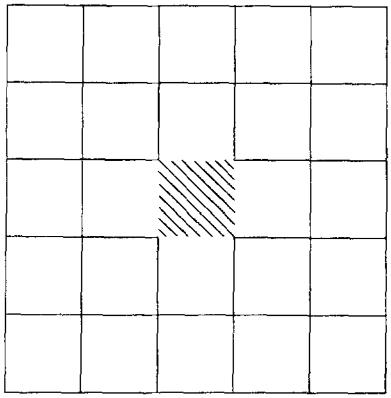

Fig. 1. One construction step with $b = 5$ and one eliminated subsquare. Details are given in the text.

a "full square" (i.e., a square whose four sides have not been eliminated) we obtain 20 full subsquares and four semi-open subsquares (small squares that have only three sides each). At the next iteration we have both full squares and semi-open squares to start with. In general, the result of the $(n+1)$ decoration depends only on the shapes' statistics after the nth iteration, and not on previous decoration steps. Thus the equation describing the division of, say, a full square into smaller subsquares is the same on any length scale. Similarly, a semi-open square yields under a decoration iteration 15 full subsquares and nine semi-open subsquares. The linear process is described by a 2×2 TMF of the form

$$G_1 = \begin{pmatrix} 20 & 15 \\ 4 & 9 \end{pmatrix} \tag{2}$$

This TMF operates on a vector at the nth step, yielding a new vector describing a smaller $(n+1)$ scale. In this notation the vector $\binom{2}{1}$ in the two-dimensional shape space describes a configuration consisting of two full squares and one semi-open square. The eigenvalues of G_1 are $\lambda_1 = 24$ and $\lambda_2 = 5$. It turns out that one can define a series of fractal dimensionalities,

$$D_i \equiv \log \lambda_i / \log b, \qquad \lambda_i > 0 \tag{3}$$

D_1 is the *primary fractal dimensionality* (associated with the largest eigenvalue) and its value, for the present example, coincides with the value calculated in Eq. (1). Besides D_1, additional *secondary fractal dimensionalities* are needed to describe the fractal. In the present example $D_2 = \log 5/\log 5 = 1$ represents a one-dimensional subset, hidden in our construction. The (nonnormalized) right eigenvector associated with λ_1, $\binom{15}{9}$), offers a straightforward geometric interpretation: the asymptotic ratio between full squares and semi-open squares is 15:4, since after many iterations this eigenvector dominates the decoration process. The fact that the second (right) eigenfector, $\binom{1}{-1}$, has a negative element means that the corresponding (one-dimensional) subset appears only in linear combinations with other sets.

Our second example is again a Sierpinski carpet ($b = 5$) with nine subsquares (including their boundaries) eliminated at each step. The shape space is now three dimensional, including in addition to full and semi-open squares also doubly open squares. The TMF in this example is

$$G_2 = \begin{pmatrix} 4 & 2 & 0 \\ 12 & 11 & 10 \\ 0 & 3 & 6 \end{pmatrix} \tag{4}$$

with eigenvalues $\lambda_1 = 16$, $\lambda_2 = 5$, $\lambda_3 = 0$. The fact that the TMF is now singular ($\lambda_3 = 0$) is related to the fact that the lattice's order of ramification[2,3] is finite.

These two examples illustrate few of the remarkable properties of TMFs. Quite generally the eigenvalues of these asymmetric nonnegative matrices are real, nonnegative integers. We now discuss some further general properties of TMFs:

(a) *Generalizability.* The construction of TMFs is quite general. This method may be employed to study planar lattices as well as lalttices in higher Euclidean dimensionalities, shapes with general boundaries, both equilibrium and nonequilibrium structures, etc. Variants of this approach allow the distinction between shapes that are topological equal but differ, e.g., in their orientation in space.

(b) *Randomization.* The process described above may be randomized. One distinguishes between *weak randomization*, when, e.g., at each iteration step exactly one subsquare is eliminated at random, and *strong randomization* when the number of eliminated subsquares may vary at each iteration, keeping only the average number fixed.

When considering a random process, an average should be taken over a product (of a large number) of TMFs. Alternatively, one can study the average TMF. This latter type of averaging is relevant when different parts of a large lattice are iterated independently.

(c) *Fractal families.* Two lattices belong to the same fractal family if their TMFs are constructed in the same shape space, and their λ_1 (or D_1) are identical. The eigenvalues and eigenvectors of TMFs that belong to the same fractal family satisfy nontrivial relations, discussed in Ref. 1.

(d) *Interpretation of the secondary fractal dimensionalities.* In various examples the secondary fractal dimensionalities have been associated (depending on the example) with pointlike, one-, or higher-dimensional subsets hidden in the fractal lattices, with various "cutting" curves, and with the fractal dimensionality of the backbone.

(e) *Mathematical formulation.* Some of the remarkable mathematical properties of nonrandom TMFs have been proven rigorously in Ref. 1. Other properties, as well as the characterization of random TMFs, still remain as general (mathematically unproven) observations.

It is quite tempting to apply this approach to various physical problems. Indeed, recently the TMF method was employed in a numerical study of percolation on a two-dimensional square lattice. The averaged TMF was constructed and analyzed, both when vacant squares were included in the statistics (the TMF was then a 6×6 matrix) or were thrown away. Some of the eigenvalues were associated with the Euclidean dimensionality of the lattice and its cutting curves, the fractal dimensionality of the infinite cluster (we obtained a value of $\simeq 1.90$), and its finite order of ramification (the latter corresponds to a zero eigenvalue). Geometrical understanding of the other eigenvalues is still awaiting further theoretical progress as well as a better numerical evaluation of the TMF.

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